

Theme: The fundamental solution of the fractional diffusion equation

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Abstract: This paper provides information on the fundamental solution of the fractional diffusion equation.

Keywords: diffusion equation, Koshi issue, fundamental solution, Fourier transform, Mittag-Leffler function.

$$\begin{cases} {}^K D_{0+t}^\alpha y(x) + \lambda y(x) = f(x) & x \in \mathbb{R}, t > 0 \\ y(0) = y_0 \end{cases} \quad (1) \rightarrow \text{Koshi (1) issue}$$

$y(x) = y_0 E_\alpha(-\lambda x^\alpha) + \int_0^x y^{\alpha-1} E_{\alpha,\alpha}(-\lambda y^\alpha) f(x-y) dy \quad (2) \rightarrow$ The solution to the Koshi(1) problem.

We derive the fundamental solution of the fractional diffusion equation by the solution of the Koshi(1) problem.

The following question is posed:

$$\begin{cases} {}^K D_{0+t}^\alpha U(x, t) - U_{xx} = f(x, t) & x \in \mathbb{R}, t > 0 \\ U(x, 0) = \varphi(x) \end{cases} \quad (3)$$

(3) We use Fourier integral substitution to find the solution of the equation.

Fourier substitution of the function f in $\mathbb{R}$

$$F[f] = F[f(x)](\xi) = \widehat{f}(\xi) := \int_{-\infty}^{+\infty} e^{ix\xi} f(x) dx, \xi \in \mathbb{R} \quad \text{appears.}$$

We use the integral Fourier substitution for a fractional order product meaning Caputo:

$$F[{}^K D_{0+t}^\alpha U] = {}^K D_{0+t}^\alpha \widehat{U}$$

We also express U_{xx} by the Fourier integral substitution:

$$\begin{aligned} F[U_{xx}] &= \int_{-\infty}^{+\infty} e^{ix\xi} U_{xx} dx = \int_{-\infty}^{+\infty} e^{-ix\xi} d(U_x) = \\ &= e^{-ix\xi} U_x \Big|_{-\infty}^{+\infty} + \xi \int_{-\infty}^{+\infty} e^{-ix\xi} U_x dx = 0 + \xi \int_{-\infty}^{+\infty} e^{-ix\xi} dU = \xi (e^{ix\xi} U \Big|_{-\infty}^{+\infty} + \\ &+ \xi \int_{-\infty}^{+\infty} e^{ix\xi} U dx = \xi (0 + i\xi \widehat{U}) = -\xi^2 \widehat{U} \end{aligned}$$

(When it comes to solutions $\int u dv = uv - \int v du$ from the fractional integration formula,

$$e^{-ix} = \cos x - i \sin x, \quad \lim_{|x| \rightarrow \infty} (U_x; U) = 0 \quad \text{we used the equations.)}$$

$$F[f(x,t)](\xi) = \hat{f}(\xi, t)$$

If we put the resulting expressions in equation(3), this equation takes the following form:

$$\begin{cases} {}^K D_{0+t}^\alpha \hat{U} + \xi^2 \hat{U} = \hat{f}(\xi, t) & \xi \in R, t > 0 \\ \hat{U}(\xi, 0) = \hat{\varphi}(\xi) \end{cases} \quad (4)$$

The view of the equation is consistent with Koshi(1) problem:

$$\begin{cases} {}^K D_{0+t}^\alpha y(x) + \lambda y(x) = f(x) & x \in R, t > 0 \\ y(0) = y_0 \end{cases} \quad (1)$$

To find the solution of equation(4), we use the solution of Koshi problem(1)

$y(x) = y_0 E_\alpha(-\lambda x^\alpha) + \int_0^x y^{\alpha-1} E_{\alpha,\alpha}(-\lambda y^\alpha) f(x-y) dy$ expression(1) is solution of equation.

Accordingly, we give the solution of equation(4):

$$\hat{U}(\xi; t) = E_\alpha(-\xi^2 t^\alpha) + \int_0^t y^{\alpha-1} E_{\alpha,\alpha}(-\xi^2 y^\alpha) \hat{f}(\xi; t-y) dy \quad (5)$$

$$\text{According to this equation } \begin{cases} {}^K D_{0+t}^\alpha U(x, t) - U_{xx} = f(x, t) & x \in R, t > 0 \\ U(x, 0) = \varphi(x) \end{cases} \quad (3)$$

we use the inverse Fourier substitution to achieve the solution of equation(3).

The inverse Fourier substitution of the function $f(x)$ is called the following integral:

$$F^{-1}[f] = F^{-1}[f(x)](\xi) = \frac{1}{2\pi} \int_R e^{ix\xi} f(\xi) d\xi$$

$$\text{Ergo : } U = F^{-1}[\hat{U}] = \frac{1}{2\pi} \int_R e^{ix\xi} \hat{U}(\xi; t) d\xi \quad (6)$$

Substituting (5) for $\hat{U}$ in equation(6), we obtain a fundamental solution of the fractional diffusion equation of the form:

$$U(x, t) = \frac{1}{2\pi} \int_R e^{ix\xi} \left[E_\alpha(-\xi^2 t^\alpha) + \int_0^t y^{\alpha-1} E_{\alpha,\alpha}(-\xi^2 y^\alpha) \hat{f}(\xi; t-y) dy \right] d\xi \quad (7)$$

Equation(7) is a solution of equation(3).

We consider that this solution is appropriate for the following issue:

$$\begin{cases} {}^K D_{0+t}^\alpha U(x, t) - U_{xx} = f & x \in R, t > 0 \\ U(x, 0) = \delta(x) \end{cases} \quad (8)$$

Here $\delta(x)$ is Dirac's delta function:

$$\delta(x - x_0) = \begin{cases} +\infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$$

$$\int_R \delta(x - x_0) dx = 1, \quad \forall y \in D(R) = C_0^\infty(R)$$

$$\int_R \delta(x - x_0) \varphi(x) dx = \varphi(x_0) \quad (9), \quad \delta(x) = \delta(-x) \text{ (dual function)}$$

Find $U(x, 0)$ by equation(7):

$$U(x, 0) = \frac{1}{2\pi} \int_R e^{ixz} \left[E_\alpha(-z^2 t^\alpha) + \int_0^0 y^{\alpha-1} E_{\alpha, \alpha}(-z^2 y^\alpha) \hat{f}(z; 0 - y) dy \right] dz = \\ \frac{1}{2\pi} \int_R e^{ixz} dz = \delta(x)$$

(This is the result $\int_0^0 y^{\alpha-1} E_{\alpha, \alpha}(-z^2 y^\alpha) \hat{f}(z; 0 - y) dy = 0$,

$$E_\alpha(-z^2 t^\alpha) = \sum_{n=0}^{\infty} \frac{(-z^2 t^\alpha)^n}{\Gamma(\alpha n + 1)} \text{ originated according to the expressions})$$

According to formula (9) it follows that the solution expressed by equation(7) is appropriated for the problem of the form(8)

$$U(x, 0) = \int_R \varphi(y) \left[\frac{1}{2\pi} \int_R e^{iz(x-y)} dz \right] dy = \int_R \varphi(y) \delta(x - y) dy = \varphi(x)$$

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