APPROXIMATE SOLUTION FOR LINEAR INEGRO-DIFFERENTIAL EQUATION OF ORDER ONE BY LEGENDRE POLYNOMIALS

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ABSTRACT. Aprroximate solution of linear integro differential equations (IDEs) of order one is presented based on the truncated series of Legendre polynomials. Reduction technique is applied to transform the IDEs into integral equations (IEs). Gauss Legendre quadrature formula is implemented to the kernel integrals and collocation method is used to fprm a system of linear algebraic equations . The collocation points are chosen as the roots of Legendre polynomials. The existence and uniqueness of the solution are shown. Rate of convergence of the proposed method is proved.

1. INTRODUCTION

There are many problems in different fields of fundamental sciences and engineering can be modeled into functional equations such as linear, nonlinear integral equations (IEs) ([2] [4] [5] [9] [17] [18] [24] , singular integral equations (SIEs)([8] [10] [15]) , partial differential equations (PDEs)([11] [12] [13] [20]), linear andnonlinear integro-differential equations(IDEs)([3] [20] [6] [7] [19] [21] [22]). The difficulty comes when we ned to find the exact solution of the equatins.

Many methods have been developed to solve linear and nonlinear IEs numerically for example for linear IEs: Operational matrix of integration method (Babolian and Fattahzadeh [2]), repeated trapezoidal method and repeated Simpson's $\frac{1}{3}$ method (Majeed and Om- ran [17]), Legendre polynomials approach (Maleknejad et al. [18]), for nonlinear IEs: homotopy perturbation method (HPM) (Biazar and Ghazvini [4]), Modified decomposition method (Bildik and Inc, [5]), Newton-Kantorovich method (Eshkuvatov et al. [9]), coupled fixed point theorem (Zhang and Chen [24]), for SIEs: Gauss-Radau and Lobatto-Jacobi direct quadrature methods (Ioakimidis [15]), collocation method (Elliot [8]), Chebyshev polynomials method (Eshkuvatov et al. [10]), for PDEs: Application of HPM for nonlinear coupled systems of reaction-diffusion equations respectively heat transfer equations (Ganji and Sadighi [12] respectively Ganji [11]), application of HPM and variational iteration methods (VIM) to nonlinear heat transfer and porous media equations (Ganji and Sadighi [13]), HPM for explicit solutions of Helmholtz equation and fifth-order KdV equation (Rafei and Ganji [20]), and so on.

Finding numerical solutions of Fredholm-Volterra IDEs is one of the oldest problems in applied mathematics. Numerous works have been focusing on the development of more advanced and effi-cient methods for solving IDEs such as differential transform method (Darania and Ebadian [6]), trapezoidal rule (Day [7]), collocation method based on Lagrange polynomials (Mustafa and Muhammad [19]), block pulse functions and its operational matrices (Rahmani et al. [21]), trigonometric scaling functions (Safdari and Aghdam [22]),

Fixed point techniques and Schauder bases (Berenguer et al. [3]), decomposition method (Adomian [1]), improved HPM (Yusufoglu [23] Reference source not found.), and references therein.

In this work, we consider Fredholm-Volterra integro-differential equations of the order one in general form

$$\sum_{i=0}^{1} b_i(s)y^{(i)}(s) = g(s) + \mu_1 \int_a^b H_1(s,t)y(t)dt + \mu_2 \int_a^s H_2(s,t)y(t)dt,$$

$y(a) = y_0$ (1)

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where $g(s), b_0(s), b_1(s) \neq 0, s \in [a, b]$ and $H_k(s,t), k \in \{1,2\}$ are know continuous functins defined on $D = \{(s,t) : a \leq s, t \leq b\}, \mu_k, k \in \{1,2\}$ ae real constants , $y(s)$ is an unknown function to be determined satisfiying inital condition.

2. Preliminaries

2.1. Legendre polynomials

One of the most frequently used polynomials in approximation theory is . Legendre polynomials Pn(r). It is orthogonal with the weight w (s)= 1 on the interval [-1,1] and can be generated (Kythe and Schaferkotter) [16] by :

(1) Rodrigues formula

$$P_n(r) = \frac{1d^n}{2^n!ds^n}(r^2-1)^n, P_o(r) = 1$$

(2) Series form

$$P_n(r) = \frac{1}{2^n}\sum_{i=1}^{[n/2]}(-1)^i\binom{n}{i}\binom{2n-2i}{n}r^{n-2i}$$

(3)Three term recurrence relations

$$P_0(r) = 1, P_1(r) = r,$$

$$P_{n+1}(r) = \frac{2n+1}{n+1}rP_n(r) - \frac{n}{n+1}P_{n-1}(r), n \geq 1$$

2.2.Gauss-Legendre Quadrature Formula

A high accurate method to approximate the integration called Gauss-Legendre quadrature formula (QF) is stated by Kyte and Schaferkotter [16] in the form

$$\int_a^b f(\tau)d\tau = \frac{b-a}{2}\sum_{j=1}^{n+1}\omega_j f(\tau_j) + R_{n+1},$$

where

$$\omega_j = \frac{2}{(1-r_j^2[p'_{n+1}(r_j)]^2}, \sum_{j=1}^{n+1}\omega_j = 2,$$

The quadrature points are $\tau_j = \frac{b-a}{2}r_j + \frac{b+a}{2}$, where r_j are the roots of the Legendre polynomial $P_{n+1}(r)$,i.e

$$P_{n+1}(r_j) = 0, j = 0, 1, ..., n$$

3. Deseription of the Method

In this section, we describe the approximate solution of Fredholm-Volterra IDEs(1). To do this, let us rewrite Equation (1) in the form

$$y'(t) = \frac{g(t)}{b_1(t)}y(t) + \mu_1\int_a^b \frac{H_1(t,\tau)}{b_1(t)}y(\tau)d\tau + \mu_2\int_a^t \frac{H_2(t,\tau)}{b_1(t)}y(\tau)d\tau, y(a) = y_0$$

set

$$Q_1(s,\tau) = \int_a^s \frac{H_1(t,\tau)}{b_1(t)}dt, Q_2(s,\tau) = \int_t^s \frac{H_2(t,\tau)}{b_1(t)}dt, f(s) = y_0 + \int_a^s \frac{g(t)}{b_1(t)}dt.$$

Choosing collocation points in the form

$$s = s_k = \frac{b-a}{2}r_k + \frac{b+a}{2}, k = 1, 2, 3, ..., n+1$$

where$r_k \in (-1,1)$ are roots of the Legendre polynomials satisfying the Equation,transforms Equation into a system of algebraic equation

$$B_nC_n = F_n$$

where

$$C_n = (c_0, c_1, ..., c_n)^T, F_n = (f(s_1), f(s_2), ..., f(s_{n+1}))^T,$$

$$B_n = \begin{pmatrix} \xi_0(s_1) & \xi_1(s_1) & \cdots & \xi_n(s_1) \\ \xi_0(s_2) & \xi_1(s_2) & \cdots & \xi_n(s_2) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_0(s_{n+1}) & \xi_1(s_{n+1} & \cdots & \xi_n(s_{n+1}) \end{pmatrix}$$

$$\xi_l(s_k) = P_l(s_k) - \mu_1 \frac{b-a}{2} \sum_{j=1}^{n+1} \omega_j Q_1(s_k, \tau_{1j}) P_l(\tau_{1j}) - \mu_2 \frac{s_k - a}{2} \sum_{j=1}^{n+1} \omega_j Q_2(s_k, \tau_{2j}) + \frac{s_k - a}{2} \sum_{j=1}^{n+1} \omega_j \frac{b_0(\tau_{2j})}{b_1(\tau_{2j})} P_l(\tau_{2j}).$$

The unknown Legendre coefficients are computed by solving Equation and the solution of Equation is estimated using $y_n(s)$ which is defined as Equation.

The sought function y(s) is estimated as follows

$$y(s) \approx y_n(s) = \sum_{l=0}^{n} c_l P_l(s), s = \frac{b-a}{2} r + \frac{b+a}{2}, r \in [-1, 1],$$

$$\tau_{1j} = \frac{b-a}{2} r_j + \frac{b+a}{2}, \tau_{2j} = \frac{s-a}{2} r_j + \frac{s+a}{2}$$

and

$$Q_{1n}(s, \tau) \approx \frac{s-a}{2} \sum_{j=1}^{n+1} \omega_j \frac{H_1(t_{1j}, \tau)}{b_1(t_{1j})}, t_{1j} = \frac{s-a}{2} r_j + \frac{s+a}{2},$$

$$Q_{2n}(s, \tau) \approx \frac{s-\tau}{2} \sum_{j=1}^{n+1} \omega_j \frac{H_2(t_{2j}, \tau)}{b_1(t_{2j})}, t_{2j} = \frac{s-\tau}{2} r_j + \frac{s+\tau}{2},$$

$$f_n(s) \approx y_0 + \frac{s-a}{2} \sum_{j=1}^{n+1} \omega_j \frac{g(t_{1j})}{b_0(t_{1j})}.$$

Convergence of the proposed method is given in the following theorem.

THEOREM *Let* $b_0(t), b_1(t), g(t) \in C^{2n+2}[a, b]$ *and* $\frac{\partial^{2n+2}}{\partial^{2n+2}} Q_i(s, t) \in C[a, b], i \in \{1, 2\}$. *Then the following estimation is true*

$$||y_n - y|| \leq C[\frac{(b-a)^{2n+3}}{(2n+3)} \left(\frac{(n+1)!}{(n+2) \cdot ... \cdot (2n+2)} \right)^2 \frac{h^{(2n+2)} + T^{(2n+2)}}{(2n+2)!}],$$

where

$$h^{(q)} = \sum_{k=0}^{q} \binom{q}{k} G^{(K)} B^{(k)}, T(q) = |\mu_1| T_1^{(q)} + |\mu_2| T_2^{(q)} + T_3^{(q)},$$

with

$$G^{(k)} = \max_{a \leq t \leq b} |\frac{d^k}{dt^k} g(t)|, \quad B^{(k)} = \max_{a \leq t \leq b} |\frac{d^{(k)}}{dt^k} \frac{1}{b_0(t)}|,$$

$$T_i^{(q)} = \sum_{k=0}^{q} \binom{q}{k} Y_{(k)} M_{ti}^{(q-k)}, i \in \{1, 2\},$$

$$T_3^{(q)} = \sum_{k=0}^{q} \binom{q}{k} Y_{(k)} M_{t3}^{(q-k)},$$

and

$$(\tau)|, \quad M_{ti}^{(q)} = \max_{a \leq s, \tau \leq b} |\frac{\partial^q}{\partial \tau^q} Q_i(s, \tau)|, \quad M_{t3}^{(q)} = \max_{a \leq t \leq b} |\frac{\partial^q}{\partial t^q} (-\frac{b_0(t)}{b_1(t)}|$$

4. 6.CONCLUSION

In this work, we have used Gauss-Legendre QF and reduction technique to solve problems (1) on the interval[a,b].Efficient method base on Legendre polynomial is presented to solve the linear Fredholm Volterra IDEs(1).convergence of the proposed method is shown in the smooth class of function.

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